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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In this paper, the authors obtained asymptotic expressions for the null and nonnull distributions of the likelihood ratio test statistic for multiple independence when the underlying distribution is complex multivariate normal. The authors also derived asymptotic null distribution of the likelihood ratio test statistic for homogeneity of the covariance matrices of several complex multivariate normal populations; asymptotic expressions are also obtained

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for the nonnull case when we have two populations. The expressions obtained in this paper are in terms of beta series.

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ASYMPTOTIC DISTRIBUTIONS OF THE LIKELIHOOD RATIO TEST STATISTICS FOR COVARIANCE STRUCTURES OF THE COMPLEX MULTIVARIATE NORMAL DISTRIBUTION*

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September 1981

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1. INTRODUCTION

The problems of testing the hypotheses on the structures of the covariance matrices of the real multivariate normal populations received considerable attention in recent years. But, not much work was done on the covariance structures of the complex multivariate normal populations. Investigations on covariance structures of the complex multivariate normal populations have important applications in the area of inference on multiple time series since certain estimates of the spectral density matrix of the multivariate stationary Gaussian time series are approximately distributed as the complex Wishart matrix. The object of this paper is to investigate the null and nonnull asymptotic distributions of the likelihood ratio statistics for testing the hypotheses on the covariance structures of the complex multivariate normal populations. For some discussions on the applications of the complex multivariate distributions, the reader is referred to Brillinger (1974), Hannan (1970) and Krishnaiah (1976).

In Section 3 of this paper, we derived the asymptotic null distribution of the likelihood ratio statistic for multiple independence whereas Section 4 is devoted to the corresponding distribution in the nonnull case under certain alternatives. An expression is derived in Section 5 for the asymptotic null distribution of the likelihood ratio statistic for multiple homogeneity of the covariance matrices of the complex multivariate normal populations. In Section 6, we derived an ex-

pression for the asymptotic nonnull distribution of the likelihood ratio test statistic for homogeneity of the covariance matrices of two complex multivariate normal populations under certain alternatives. The hypotheses considered in Sections 3 - 6 arise in studying certain linear structures of the covariance matrices. For a discussion of these problems in the real case, the reader is referred to Krishnaiah and Lee (1974). The expressions obtained in this paper are in terms of the beta series. In the null cases, it is found that the accuracy of the approximations based on the first terms of the asymptotic series are sufficient for many practical purposes. Krishnaiah, Lee and Chang (1976) approximated the null distributions of certain powers of the likelihood ratio test statistics for multiple independence and multiple homogeneity of the covariance matrices of the complex multivariate normal populations with Pearson's Type I distributions. But these approximations are based upon empirical investigations, whereas the investigations in the present paper are analytic in nature. In the real case, Rao (1951) gave a useful approximation, in terms of beta series, for the null distribution of certain power of the multivariate beta matrix.

2. PRELIMINARIES

In this section, we define some notation and give some lemmas which are needed in the sequel.

Let f(x) be a function of the real variable x defined for x > 0. Then, the Mellin's integral transform of f(x) is defined (e.g., see Titchmarch (1937)) as

$$M\{f(\cdot)|t\} = \int_0^\infty x^{t-1} f(x)dx \tag{2.1}$$

where t is a complex variate. If f(x) is absolutely continuous in (0,1), then

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} M\{f(\cdot)|t\} x^{-t} dt. \tag{2.2}$$

Note that when $f(x) = (1-x)^{g-1}$, $0 < x < 1$,

$$M\{f(\cdot)|t\} = \int_0^1 x^{t-1}(1-x)^{g-1}dx = \frac{\Gamma(t)\Gamma(g)}{\Gamma(t+g)}$$

for Real (t) > 0 and Real (g) > 0. Hence

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(t)}{\Gamma(t+g)} x^{-t} dt = \frac{(1-x)^{g-1}}{\Gamma(g)}, \; c > 0. \tag{2.3}$$

The following lemma was proved by Nair (1940):

Lemma 2.1. Let $\phi(t) = \int x^t p(x)dx$ where p(x) is the density of the random variable x. If

$$\phi(t) = O(t^{-\nu}) \tag{2.4}$$

with Real (t) tending to infinity, then $\phi(t)$ can be expanded as

$$\phi(t) = \sum_{i=0}^{\infty} R_i \Gamma(t+a)/\Gamma(t+a+\nu+i) \tag{2.5}$$

where a is any constant.

Lemma 2.2.

Consider the series $\sum_{i=1}^{\infty} \alpha_i x^i$ which converges to the function g(x) in the neighborhood of x=0 (or the asymptotic expansion of g(x) when x=0). Then

$$\exp\{g(x)\} = 1 + \sum_{i=1}^{\infty} \beta_i x^i \tag{2.6}$$

where β_j's satisfy the recurrence relation

$$\beta_j = \frac{1}{j} \sum_{k=1}^{j} k\alpha_k \beta_{j-k}, \quad \beta_0 = 1. \tag{2.7}$$

We use the following notations as defined in James (1964). The complex multivariate gamma functions $\tilde{\Gamma}_p(a)$ and $\tilde{\Gamma}_p(a,\kappa)$ are given by

$$\tilde{\Gamma}_p(a) = \pi^{p(p-1)/2} \prod_{i=1}^{p} \Gamma(a-i+1). \tag{2.8}$$

$$\tilde{\Gamma}_p(a,\kappa) = \pi^{p(p-1)/2} \prod_{i=1}^{p} \Gamma(a-i+1+k_i) \tag{2.9}$$

where $\kappa = (k_1, k_2, \ldots, k_p)$ is a partition of the integer k such that $k_1 \geq \ldots \geq k_p \geq 0$ and $k = k_1 + \ldots + k_p$. The transpose and conjugate of a complex matrix B are denoted by B' and $\bar{B}$ respectively. Also, let $\tilde{C}_\kappa(A)$ denote the zonal polynomial of a Hermitian matrix A, (i.e., $A = \bar{A}'$)

Lemma 2.3. For any integer r, variate x and Hermitian positive definite V, we have

$$\sum_{k=r}^{\infty} \sum_{\kappa} \frac{x^k \tilde{C}_\kappa(V)}{(k-r)!} = x^r \ (\mathrm{tr}\ V)^r \ \mathrm{etr}\ (xV) \tag{2.10}$$

$$\sum_{k=0}^{\infty} \sum_{\kappa} \frac{x^k \tilde{a}_1(\kappa) \tilde{C}_\kappa(V)}{k!} = (x^2 \mathrm{tr}\ V^2 - x\ \mathrm{tr}\ V)\ \mathrm{etr}(xV) \tag{2.11}$$

$$\begin{aligned}\sum_{k=r}^{\infty} \sum_{\kappa} \frac{x^k \tilde{a}_1(\kappa) \tilde{C}_\kappa(V)}{(k-r)!} = \{&x^{r+2} \mathrm{tr}\ V^2 (\mathrm{tr}\ V)^r - x^{r+1} (\mathrm{tr}\ V)^{r+1} \\ &+ 2r\ x^{r+1}\ \mathrm{tr}\ V^2 (\mathrm{tr}\ V)^{r-1} - r x^r (\mathrm{tr}\ V)^r \\ &+ r(r-1) x^r \mathrm{tr}\ V^2 (\mathrm{tr}\ V)^{r-2}\} \mathrm{etr}(xV)\end{aligned} \tag{2.12}$$

$$\begin{aligned}\sum_{k=0}^{\infty} \sum_{\kappa} \frac{x^k \tilde{a}_1^2(\kappa) \tilde{C}_\kappa(V)}{k!} = \{&x^4 (\mathrm{tr}\ V^2)^2 + 4x^3\ \mathrm{tr}\ V^3 - 2x^3\ \mathrm{tr}\ V\ \mathrm{tr}\ V^2 \\ &+ 3x^2 (\mathrm{tr}\ V)^2 - 4x^2\ \mathrm{tr}\ V^2 + x\ \mathrm{tr}\ V\}\ \mathrm{etr}(xv)\end{aligned} \tag{2.13}$$

$$\begin{aligned}\sum_{k=0}^{\infty} \sum_{\kappa} \frac{x^k \tilde{a}_2(\kappa) \tilde{C}_\kappa(V)}{k!} = \{&2x^3\ \mathrm{tr}\ V^3 + 3x^2 (\mathrm{tr}\ V)^2 - 3x^2 \mathrm{tr}\ V^2 \\ &+ 2x\ \mathrm{tr}\ V\}\ \ \mathrm{etr}(xV)\end{aligned} \tag{2.14}$$

where

$$\tilde{a}_1(\kappa) = \sum_{j=1}^{p} k_j (k_j - 2j)$$

$$\tilde{a}_2(\kappa) = 2 \sum_{j=1}^{p} k_j (k_j^2 - 3jk_j + 3j^2) \tag{2.15}$$

The above lemma was proved in Hayakawa (1972). We need the following in the sequel.

From Hayakawa (1972), Lemma 1

$$(\tilde{a}_1(\kappa)+k)\tilde{C}_\kappa(V) = \operatorname{tr}(\Lambda\partial)^2\tilde{C}_\kappa(V)|_{V=\Lambda} \tag{2.16}$$

$$\begin{aligned}&\{3\tilde{a}_1^2(\kappa)-2\tilde{a}_2(\kappa)+6k\tilde{a}_1(\kappa)-6\tilde{a}_1(\kappa)+3k^2-2k\}\tilde{C}_\kappa(V)\\ &= [8(\operatorname{tr}(\Lambda\partial)^3)+3(\operatorname{tr}(\Lambda\partial)^2)^2]\tilde{C}_\kappa(V)\ |_{V=\Lambda}\end{aligned} \tag{2.17}$$

where Λ is a diagonal matrix of eigenvalues of V, ∂ is the Hermitian differential operator matrix

$$\partial = \partial_R+i\partial_I,\ \partial = (\partial_{\alpha\beta}),\ \partial_R = (\partial^R_{\alpha\beta}),\ \partial_I = (\partial^I_{\alpha\beta})$$

and

$$\partial^R_{\alpha\beta} = \frac{(1+\delta_{\alpha\beta})}{2}\frac{\partial}{\partial V^R_{\alpha\beta}},\quad \partial^I_{\alpha\beta} = \frac{(1-\delta_{\alpha\beta})}{2}\frac{\partial}{\partial V^I_{\alpha\beta}}$$

where

$$V = (V_{\alpha\beta}) = (V^R_{\alpha\beta}+iV^I_{\alpha\beta}),\ \delta_{\alpha\beta} = \begin{cases}1, & \alpha=\beta\\ 0, & \alpha\neq\beta\end{cases}.$$

By definition of zonal polynomial

$$\sum_\kappa \tilde{C}_\kappa(V) = (\operatorname{tr} V)^k. \tag{2.18}$$

and Eq. (2.16)

$$\begin{aligned}\sum_\kappa \tilde{a}_1(\kappa)\tilde{C}_\kappa(V) &= \operatorname{tr}(\Lambda\partial)^2(\operatorname{tr} V)^k|_{V=\Lambda} - k(\operatorname{tr}V)^k\\ &= k(k-1)\operatorname{tr} V^2(\operatorname{tr} V)^{k-2}-k(\operatorname{tr} V)^k.\end{aligned} \tag{2.19}$$

Multiply both sides of Eq. (2.16) by $\tilde{a}_1(\kappa)$ and sum over κ, by applying Eq. (2.19), we have

$$\sum_{\kappa} \tilde{a}_1^2(\kappa)\tilde{C}_\kappa(V) = (3k^2-2k)(\mathrm{tr}\ V)^k - 2k^2(k-1)\ \mathrm{tr}\ V^2(\mathrm{tr}\ V)^{k-2}$$
$$+ 4\,k(k-1)(k-2)\ \mathrm{tr}\ V^3(\mathrm{tr}\ V)^{k-3}$$
$$+ k(k-1)(k-2)(k-3)(\mathrm{tr}\ V^2)^2(\mathrm{tr}\ V)^{k-4}\,. \qquad (2.20)$$

From Eq. (2.17) and Eqs. (2.18), (2.19), (2.20), we obtain

$$\sum_{\kappa} \tilde{a}_2(\kappa)\tilde{C}_\kappa(V) = (3\,k^2-k)(\mathrm{tr}\ \Lambda)^k - 3k(k-1)\ \mathrm{tr}\ V^2(\mathrm{tr}\ V)^{k-2}$$
$$+ 2\,k(k-1)\,(k-2)\ \mathrm{tr}\ V^3\ (\mathrm{tr}\ V)^{k-3}\,. \qquad (2.21)$$

Eqs. (2.18) - (2.21) are needed subsequently. Here we note that Lemma 2.3 follows by multiplying both sides of Eqs. (2.18) - (2.21) by $\frac{x^k}{k!}$ and summing over k.

3. ASYMPTOTIC DISTRIBUTION OF THE LIKELIHOOD RATIO STATISTIC FOR TESTING MULTIPLE INDEPENDENCE

Let $\underset{\sim}{Z}' = (\underset{\sim}{Z}_1', \ldots, \underset{\sim}{Z}_d')$ be distributed as a complex multivariate normal with mean vector $\underset{\sim}{\mu}'$ and covariance matrix Σ. Also, let $E\{(\underset{\sim}{Z}_i - \underset{\sim}{\mu}_i)(\bar{\underset{\sim}{Z}}_j - \bar{\underset{\sim}{\mu}}_j)'\} = \Sigma_{ij}$. We will assume that $\underset{\sim}{Z}_i$ is of order $p_i \times 1$ and $b = p_1 + \ldots + p_d$. Consider the hypothesis H_o where

$$H_o: \Sigma_{ij} = 0 \quad (i \neq j = 1, \ldots, d). \tag{3.1}$$

Let $(\underset{\sim}{Z}_{1j}', \ldots, \underset{\sim}{Z}_{dj}')$, $(j = 1, 2, \ldots, N)$, be N independent observations on $\underset{\sim}{Z}'$. Also, let $A = (A_{\ell m})$, where

$$A_{\ell m} = \sum_{j=1}^{N} (\underset{\sim}{Z}_{\ell j} - \underset{\sim}{Z}_{\ell \cdot})(\bar{\underset{\sim}{Z}}_{mj} - \bar{\underset{\sim}{Z}}_{m \cdot})', \quad \underset{\sim}{Z}_{\ell \cdot} = \frac{1}{N} \sum_{j=1}^{N} \underset{\sim}{Z}_{\ell j} \; (\ell, m = 1, \ldots, d). \tag{3.2}$$

The likelihood ratio statistic for testing H_0 is

$$\lambda = |A| \; \{ \prod_{i=1}^{d} |A_{ii}| \}^{-1} . \tag{3.3}$$

Make the transformation $u = \lambda^{1/s}$, where s is a constant to be chosen to govern the rate of convergence. The h^{th} moment of u is

$$E(u^h) = \{ \prod_{j=1}^{b} \frac{\Gamma(n + \frac{h}{s} - j + 1)}{\Gamma(n - j + 1)} \} \{ \prod_{i=1}^{d} \prod_{\alpha=1}^{p_i} \frac{\Gamma(n - \alpha + 1)}{\Gamma(n + \frac{h}{s} - \alpha + 1)} \}. \tag{3.4}$$

By using the Mellin's inverse transform, the density of u becomes

$$f(u) = \frac{K(b,d,p_i,n)}{2\pi i} \int_{c-i\infty}^{c+i\infty} u^{-h-1} \frac{\prod_{j=1}^{b} \Gamma(n+\frac{h}{s} - j + 1)}{\prod_{i=1}^{d} \prod_{\alpha=1}^{p_i} \Gamma(n+\frac{h}{s} - \alpha + 1)} dh \qquad (3.5)$$

where $K(b,d,p_i,n) = \prod_{i=1}^{d} \sum_{\alpha=1}^{p_i} \Gamma(n-\alpha+1) / \prod_{j=1}^{b} \Gamma(n-j+1)$. Set $n=m+\delta, e = c + m s$ where $n = N-1$ and $m+\frac{h}{s} = \frac{t}{s}$ where δ is also a converging factor to be chosen. Then we have

$$f(u) = \frac{K(b,d,p_i,n)}{2\pi i} u^{sm-1} \int_{e-i\infty}^{e+i\infty} u^{-t} \phi(t)\, dt \qquad (3.6)$$

and

$$\phi(t) = \prod_{j=1}^{p} \Gamma(\frac{t}{s} + \delta - j + 1) / \prod_{i=1}^{d} \prod_{\alpha=1}^{p_i} \Gamma(\frac{t}{s} + \delta - \alpha + 1). \qquad (3.7)$$

By the use of formula for the asymptotic expansion of gamma function, we have

$$\log \Gamma(x+g) = \log \sqrt{2\pi} + (x+g-\frac{1}{2}) \log x - x - \sum_{r=1}^{\infty} (-1)^r \frac{B_{r+1}(g)}{r(r+1)x^r} \qquad (3.8)$$

for g bounded and $B_r(g)$ is the Bernoulli polynomial of degree r. So,

$$\log \phi(t) = \log s^{\nu} + \log t^{-\nu} + \sum_{r=1}^{\infty} \frac{A_r}{t^r} \qquad (3.9)$$

where

$$\nu = \frac{b(b+1)}{2} - \sum_{i=1}^{d} \frac{p_i(p_i+1)}{2} \qquad (3.10)$$

$$A_r = \frac{(-1)^r\, s^r}{r(r+1)} \left[\sum_{i=1}^{d} \sum_{=1}^{p_i} B_{r+1}(\delta-\alpha+1) - \sum_{i=1}^{b} B_{r+1}(\delta-i+1) \right]. \quad (3.11)$$

Hence

$$\phi(t) = s^{\nu}\, t^{-\nu} \left[1 + \sum_{r=1}^{\infty} \frac{Q_r}{t^r} \right]. \quad (3.12)$$

the coefficient Q_r can be obtained by the recursive equation

$$Q_r = \frac{1}{r} \sum_{\ell=1}^{r} \ell A_\ell\, Q_{r-\ell}, \quad Q_0 = 1. \quad (3.13)$$

Since $\phi(t) = O(t^{-\nu})$, we can use Eq. (2.5) and write $\phi(t)$ as follows:

$$t^{-\nu} \{1 + \sum_{r=1}^{\infty} \frac{Q_r}{t^r}\} = \sum_{i=0}^{\infty} R_i \frac{\Gamma(t+a)}{\Gamma(t+a+\nu+i)} \quad (3.14)$$

and a is a constant to be chosen to govern the rate of convergence for the resultant series. Using Eq. (3.8) to expand the gamma function on the right hand side of Eq. (3.14), we have

$$\log \frac{\Gamma(t+a)}{\Gamma(t+a+\nu+i)} = -(\nu+i) \log t + \sum_{j=1}^{\infty} \frac{A_{ij}}{t^j} \quad (3.15)$$

where

$$A_{ij} = \frac{(-1)^j}{j(j+1)} [B_{j+1}(\nu+a+i) - B_{j+1}(a)]. \quad (3.16)$$

Thus

$$\frac{\Gamma(t+a)}{\Gamma(t+a+\nu+i)} = t^{-(\nu+i)} \left[1 + \sum_{j=1}^{\infty} \frac{C_{ij}}{t^j} \right] \quad (3.17)$$

and C_{ij} can be recursively computed as

$$C_{ij} = \frac{1}{j} \sum_{\ell=1}^{j} \ell \, A_{i\ell} \, C_{i,j-\ell}, \quad C_{i0} = 1. \tag{3.18}$$

Substituting Eq. (3.17) in Eq. (3.14) and equating the coefficient of same powers of t, R_i is determined explicitly as

$$\sum_{j=0}^{i} R_{i-j} \, C_{i-j,j} = Q_i ; \quad R_0 = 1. \tag{3.19}$$

Now use Eqs.(3.14), (3.12) in Eq. (3.6) and note that the term by term integration is valid since a factorial series is uniformly convergent in a half-plane (Doetsch (1971)). Then we have

$$f(u) = K(b,d,p_i,n) \, u^{sm-1} \, s^{\nu} \sum_{j=0}^{\infty} R_j \, \frac{1}{2\pi i} \int_{e-i\infty}^{e+i\infty} u^{-t} \frac{\Gamma(t+a)}{\Gamma(t+a+\nu+j)} \, dt \tag{3.20}$$

Using Eq. (2.3) for the integral, we have

$$f(u) = K(b,d,p_i,n) \, s^{\nu} \sum_{j=0}^{\infty} R_j \, u^{sm+a-1} \, (1-u)^{\nu+j-1} / \Gamma(\nu+j), 0 \le u \le 1 . \tag{3.21}$$

Thus the c.d.f. of u in terms of incomplete beta functions $I_x(\cdot,\cdot)$ is

$$\mathrm{Prob}\,(u \le x) = K(b,d,p_i,n) \, s^{\nu} \sum_{j=0}^{\infty} R_j \, I_x(sm+a, \nu+j) \, \frac{\Gamma(sm+a)}{\Gamma(sm+a+\nu+j)} \tag{3.22}$$

where

$$I_x(\alpha,\beta) = \int_0^x \frac{(\alpha+\beta)}{\Gamma(\alpha)\ \Gamma(\beta)}\ u^{\alpha-1}\ (1-u)^{\beta-1}\ du. \tag{3.23}$$

Further expansion of $K(b,d,p_i,n) = \dfrac{\prod_{i=1}^{d} \prod_{\alpha=1}^{p_i} \Gamma(n-\alpha+1)}{\prod_{j=1}^{b} \Gamma(n-j+1)}$ and

$\dfrac{\Gamma(sm+a)}{\Gamma(sm+a+\nu+j)}$ gives us that

$$\log K(b,d,p_i,n) = \log m^{\nu} + \sum_{r=1}^{\infty} \frac{A_r^*}{m^r}$$

where

$$A_r^* = \frac{(-1)^r}{r(r+1)} \left[\sum_{i=1}^{b} B_{r+1}(\delta-i+1) - \sum_{i=1}^{d} \sum_{\alpha=1}^{p_i} B_{r+1}(\delta-\alpha+1) \right] = -\frac{A_r}{s^r} \tag{3.24}$$

and

$$K(b,d,p_i,n) = m^{\nu} \left[1 + \sum_{r=1}^{\infty} \frac{Q_r^*}{m^r} \right],$$

where

$$Q_r^* = \frac{1}{r} \sum_{\ell=1}^{r} \ell\ A_\ell^*\ Q_{r-\ell}^*,\quad Q_0^* = 1. \tag{3.25}$$

Also

$$\log \frac{\Gamma(sm+a)}{\Gamma(sm+a+\nu+j)} = \log\ (ms)^{-(\nu+j)} + \sum_{r=1}^{\infty} \frac{A_{jr}^*}{m^r} \tag{3.26}$$

where

$$A^*_{jr} = \frac{(-1)^r}{s^r\ r(r+1)}\,[B_{r+1}(a+\nu+j) - B_{r+1}(a)] = \frac{A_{jr}}{s^r} \tag{3.27}$$

and

$$\frac{\Gamma(sm+a)}{\Gamma(sm+a+\nu+j)} = (ms)^{-(\nu+j)}\left[1 + \sum_{r=1}^{\infty} \frac{C^*_{jr}}{m^r}\right] \tag{3.28}$$

$$C^*_{jr} = \frac{1}{r}\sum_{\ell=1}^{r} \ell\, A^*_{j\ell}\, C^*_{j,r-\ell} = \frac{C_{jr}}{s^r},\quad C^*_{j0} = 1\,. \tag{3.29}$$

Hence, Eq. (3.22) is of the form

$$\begin{aligned}\text{Prob } (u \le \delta) &= \left(1 + \sum_{r=1}^{\infty} \frac{Q^*_r}{m^r}\right) \sum_{j=0}^{\infty} R_j\ I_x(sm+a,\nu+j)(ms)^{-j} \\ &\quad \times \left(1 + \sum_{r=1}^{\infty} \frac{C_{jr}}{(sm)^r}\right) \qquad (3.30) \\ &= I_x(sm+a,\nu) + \sum_{i=1}^{\infty} \frac{1}{m^i}\, G_i\end{aligned}$$

where

$$G_i = \sum_{j=0}^{i} R_{i-j}\ I_x(sm+a,\nu+i-j) \sum_{\ell=0}^{j} \frac{Q^*_\ell\ C_{i-j,j-\ell}}{s^{i-\ell}}\,. \tag{3.31}$$

The exact c.d.f. can be calculated with the above formula when the sample size n is small. A suitable choice of δ will make m large in order to expedite the convergence of the series in Eq. (3.30).

For large sample size n, let us now examine the first few terms of Eq. (3.30). We know that

$$G_1 = I_x(sm+a,\nu)\left(\frac{C_{01}}{s} + Q_1^*\right) + R_1\ I_x(sm+a,\nu+1)/s \qquad (3.32)$$

$$G_2 = I_x(sm+a,\nu)\left(\frac{C_{02}}{s^2} + \frac{Q_1^*\ C_{01}}{s} + Q_2^*\right) + R_1\ I_x(sm+a,\nu+1)\left(\frac{C_{11}}{s^2} + \frac{Q^*{}_1}{s}\right)$$

$$+ R_2\ I_x(sm+a,\nu+2)/s^2 \qquad (3.33)$$

$$G_3 = I_x(sm+a,\nu)\left(\frac{C_{03}}{s^3} + \frac{Q_1^*\ C_{02}}{s^2} + \frac{Q_2^*\ C_{01}}{s} + Q_3^*\right)$$

$$+ R_1\ I_x(sm+a,\nu+1)\left(\frac{C_{12}}{s^3} + \frac{Q_1^* C_{11}}{s^2} + \frac{Q_2^*}{s}\right)$$

$$+ R_2\ I_x(sm+a,\nu+2)\left(\frac{C_{21}}{s^3} + \frac{Q_1^*}{s^2}\right) + R_3\ I_x(sm+a,\nu+3)/s^3 \quad (3.34)$$

From Eqs.(3.24), (3.25), we have

$$Q_1^* = A_1^* = -\frac{1}{2}\left[\sum_{i=1}^{b} B_2(\delta-i+1) - \sum_{i=1}^{d}\sum_{\alpha=1}^{p_i} B_2(\delta-\alpha+1)\right] \qquad (3.35)$$

Eqs. (3.11), (3.13) give

$$Q_1 = A_1 = -\ sA_1^* = -\ sQ_1^* \qquad (3.36)$$

From Eq. (3.19), we have

$$R_1 + C_{01} = Q_1 \qquad (3.37)$$

Eq. (3.16), (3.18) give

$$C_{01} = A_{01} = \frac{-1}{2}\,[B_2(\nu+a) - B_2(a)] \qquad (3.38)$$

By expanding $B_2(x) = x^2 - x + \frac{1}{6}$, which is the Bernoulli polynomial and setting

$$\nu_1 = \frac{b(b+1)(2b+1)}{6} - \sum_{i=1}^{d} \frac{p_i(p_i+1)(2p_i+1)}{6} \tag{3.39}$$

$$\delta_o = \frac{\nu_1 - \nu}{2\nu} \tag{3.40}$$

$$a_o = (1-\nu)/2 \tag{3.41}$$

in Eq. (3.35) and (3.38), we obtain

$$A_1^* = Q_1^* = C_{01} = 0 \tag{3.42}$$

$$A_1 = Q_1 = R_1 = 0 \ . \tag{3.43}$$

Hence these results together with Eqs. (3.11), (3.13), (3.16), (3.18), (3.19) give

$$R_2 + C_{02} = Q_2 \tag{3.44}$$

$$C_{02} = A_{02} = \frac{1}{6}\left[B_3(\nu+a_o) - B_3(a_o)\right] \tag{3.45}$$

$$Q_2 = A_2 = \frac{s^2}{6}\left[\sum_{i=1}^{d}\sum_{\alpha=1}^{p_i} B_3(\delta_o-\alpha+1) - \sum_{i=1}^{b} B_3(\delta_o-i+1)\right] . \tag{3.46}$$

By equating Eq. (3.45) and Eq. (3.46), and expanding $B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$, we obtain

$$s^2 = \frac{(1-\nu^2)\nu}{24\, A_2^*} \ . \tag{3.47}$$

If we choose s_o to be the positive root of Eq. (3.47), then

$$R_2 = 0. \tag{3.48}$$

Now

$$\frac{C_{02}}{s_o^2} = \frac{Q_2}{s_o^2} = \frac{A_2}{s_o^2} = - A_2^* = - Q_2^* \ . \tag{3.49}$$

Hence

$$\frac{C_{02}}{s_o^2} + Q_2^* = C_{02}^* + Q_2^* = 0 \tag{3.50}$$

In Eq. (3.19)

$$R_3 = Q_3 - C_{03} \tag{3.51}$$

Eqs. (3.11), (3.13) give

$$Q_3 = A_3 = - s_o^3 A_3^* = - s_o^3 Q_3^*. \tag{3.52}$$

So

$$R_3 = - s_o^3 \left(Q_3^* + \frac{C_{03}}{s_o^3}\right). \tag{3.53}$$

All these identities induce that

$$G_1 = G_2 = 0 \tag{3.54}$$

$$G_3 = \frac{R_3}{s_o^3} \left(I_x(s_o m_o + a_o, \nu+3) - I_x(s_o m_o + a_o, \nu)\right) \tag{3.55}$$

where

$$m_o = n - \delta_o \, . \tag{3.56}$$

Thus by so choosing δ_o, a_o and s_o, we have the c.d.f. of u in asymptotic form as

$$\text{Prob}\ (u \le x) = I_x(s_o m_o + a_o, \nu)$$
$$+ \frac{1}{m_o^3} \frac{R_3}{s_o^3} [I_x(s_o m_o + a_o, \nu+3)$$
$$- I_x(s_o m_o + a_o, \nu)] \ + O(m_o^{-4})\ . \qquad (3.57)$$

For $d = 2$, $p_1 = 1$, we have $\nu_o = b-1$, $a_o = \frac{2-b}{2}$, $\delta_o = \frac{b}{2}$, $s_o = 1$, and

$$\text{Prob}\ (\lambda \le x) = I_x(N-b, b-1)\ . \qquad (3.58)$$

Table 1 gives a comparison of the accuracy of the approximation by taking the first term in (3.57) with the approximation obtained by taking the first two terms in (3.57). In the table, the constant $\tilde{w}$ is defined by $c = \exp(-\tilde{w}/2)$ where $P[\lambda \le c] = \alpha$ and the values of $\tilde{w}$ are taken from the tables of Krishnaiah, Lee and Chang (1976). The value of $P[\lambda \le c]$ is denoted by α_1 or α_2 according as one term or two terms in (3.57) are used. Also, $p_i = p$ for $i = 1, 2, \ldots, d$.

TABLE 1

Significance Level Associated with the Asymptotic Expression for the Likelihood Ratio Test for Independence

n	d=3, p=1, α=0.05			d=3, p=2, α=0.05			d=5, p=2, α=0.05		
	$\tilde{w}$	α_1	α_2	$\tilde{w}$	α_1	α_2	$\tilde{w}$	α_1	α_2
10	1.459	.0499	.0499	5.143	.0488	.0493	-	-	-
15	.923	.0499	.0499	2.987	.0498	.0499	9.746	.0454	.0472
20	.675	.0500	.0500	2.113	.0499	.0500	6.510	.0486	.0493
30	.439	.0502	.0502	1.337	.0495	.0495	3.950	.0496	.0498

TABLE 1 (Continued)

n	d=3, p=1, α=0.10			d=3, p=2, α=0.10			d=5, p=2, α=0.10		
	$\tilde{w}$	α_1	α_2	$\tilde{w}$	α_1	α_2	$\tilde{w}$	α_1	α_2
10	1.233	.0998	.0999	4.679	.0982	.0991	-	-	-
15	.780	.0999	.1000	2.722	.0995	.0997	9.222	.0928	.0958
20	.570	.1004	.1004	1.926	.0998	.0998	6.167	.0979	.0990
30	.372	.0994	.0994	1.218	.0995	.0995	3.743	.0996	.0999

4. NONNULL ASYMPTOTIC DISTRIBUTION OF THE LIKELIHOOD RATIO TEST STATISTIC FOR INDEPENDENCE OF TWO SETS OF VARIABLES

Consider the case $d = 2$, $p_1 = p$ and $p_2 = q$ in Section 3. $(\underset{\sim}{Z}_1', \underset{\sim}{Z}_2')$ is distributed as complex multivariate normal with mean vector $\underset{\sim}{\mu}'$ and covariance matrix Σ where

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} . \tag{4.1}$$

Also, $A = (A_{ij})$ is as defined in Eq. (3.2) and $p \leq q$. The likelihood ratio statistic for testing $\Sigma_{12} = 0$ against the alternative $\Sigma_{12} \neq 0$ is given by

$$\begin{aligned} \lambda &= |A|/|A_{11}||A_{22}| = |I - A_{21}A_{11}^{-1}A_{12}A_{22}^{-1}| \\ &= |I - R^2| \end{aligned} \tag{4.2}$$

where $R^2 = \text{diag}\,(\gamma_1^2, \ldots, \gamma_p^2)$, $\gamma_1^2 \geq \ldots \geq \gamma_p^2$ and γ_i's are the sample canonical correlations between $\underset{\sim}{Z}_1$ and $\underset{\sim}{Z}_2$.

Let $n = N-1 = m_o + \delta_o$, where δ_o is defined in Eq. (3.40), and assume

$$P^2 = \frac{W}{m_o} \tag{4.3}$$

where W is fixed as $m_o \to \infty$. Also, $P^2 = \text{diag}\,(\rho_1^2, \ldots, \rho_p^2)$ and $\rho_1^2 \geq \ldots \geq \rho_p^2$ are the roots of the characteristic equation

$$|\Sigma_{12}\Sigma_{22}\Sigma_{21} - \rho^2\Sigma_{11}| = 0. \tag{4.4}$$

Using the density of $\gamma_1^2, \ldots, \gamma_p^2$ given in James (1964) the h^{th} moment of $u = \lambda^{1/s_o}$, s_o as defined in Eq. (3.41), becomes

$$E(u^h) = |I-P^2|^n \sum_k \sum_\kappa \frac{\tilde{C}_\kappa(P^2)}{k!} \prod_{i=1}^{p} \frac{[\Gamma(n+k_i-i+1)]^2}{\Gamma(n-i+1)\ \Gamma(n-q-i+1)}$$

$$\times \frac{\Gamma(n-q+\frac{h}{s_o}-i+1)}{\Gamma(n+\frac{h}{s_o}+k_i-i+1)} \tag{4.5}$$

where $\tilde{C}_\kappa(M)$ is the zonal polynomial of Hermitian matrix M, $\kappa = (k_1, k_2, \ldots, k_p)$ is a partition of $k, k_1 \geq k_2 \geq \ldots k_p \geq 0$, and $k = k_1 + \ldots + k_p$. The non-null distribution is obtained by using the inverse Mellin transform and by following the argument from Eq. (3.4) to Eq. (3.29). Substituting δ_o, a_o and s_o as in Eq. (3.40), (3.41) and (3.47), we have

$$\text{Prob } (u \leq x) = |I-P^2|^n \sum_k \sum_\kappa \frac{\tilde{C}_\kappa(P^2)}{k!} m_c^k \left[1 + \sum_{r=1}^{\infty} \frac{Q_r^*(k)}{m_o^r}\right]$$

$$\times \sum_{j=0}^{\infty} R_j(k)\ I_x(s_o m_o + a_o, \nu+j+k)(m_o s_o)^{-j}$$

$$\times \left[1 + \sum_{r=1}^{\infty} \frac{C_{jr}^*(k)}{m_o^r}\right]. \tag{4.6}$$

Note that

$$I_x(s_o m_o + a_o, \nu+j+k) = \int_0^x \frac{(s_o m_o + a_o + \nu + j + k)}{\Gamma(s_o m_o + a_o)\ \Gamma(\nu+j+k)}$$

$$\times\ u^{s_o m_o + a_o - 1} (1-u)^{\nu+j+k-1} du \qquad (4.7)$$

is a function of k, and we have the following expressions analogous to Eqs. (3.11), (3.13), (3.16), (3.18), (3.19), (3.24), (3.25), (3.27) and (3.29)

$$\sum_{j=0}^{i} R_{i-j}(k)\ C_{i-j,j}(k) = Q_i(k),\ R_0(k) = 1 \qquad (4.8)$$

$$Q_r(k) = \frac{1}{r} \sum_{\ell=1}^{r} \ell\, A_\ell(k)\ Q_{r-\ell}(k) \qquad (4.9)$$

$$A_r(k) = \frac{(-1)^r\ s_o^r}{r(r+1)} \sum_{i=1}^{p} [(B_{r+1}(\delta_o + k_i - i + 1) - B_{r+1}(\delta_o - q - i + 1)] \qquad (4.10)$$

$$C_{ir}(k) = \frac{1}{r} \sum_{\ell=1}^{r} \ell\, A_{i\ell}(k)\ C_{i,r-\ell}(k);\ C_{i0}(k) = 1 \qquad (4.11)$$

$$A_{ir}(k) = \frac{(-1)^r}{r(r+1)} [B_{r+1}(\nu + a_o + k + i) - B_{r+1}(a_o)] \qquad (4.12)$$

$$Q_r^*(k) = \frac{1}{r} \sum_{\ell=1}^{r} \ell\, A_\ell^*(k)\ Q_{r-\ell}^*(k), \quad Q_0^*(k) = 1 \qquad (4.13)$$

$$A_r^*(k) = \frac{(-1)^r}{r(r+1)} \sum_{i=1}^{p} [B_{r+1}(\delta_o - i + 1) + B_{r+1}(\delta_o - q - i + 1)$$

$$- 2B_{r+1}(\delta_o + k_i - i + 1)] \qquad (4.14)$$

$$C^*_{ir}(k) = \frac{1}{r} \sum_{\ell=1}^{r} \ell \, A^*_{i\ell}(k) \, C^*_{i,r-\ell}(k), \quad C^*_{i0}(k) = 1 \tag{4.15}$$

$$A^*_{ir}(k) = \frac{(-1)^r}{s_o^r \, r(r+1)} \left[B_{r+1}(a_o+\nu+k+i) - B_{r+1}(a_o)\right] \tag{4.16}$$

and ν is defined in Eq. (3.10); here

$$\nu = pq \; . \tag{4.17}$$

The asymptotic expansion of $|I - P^2|^n$ gives

$$|I - P^2|^n = \left| I - \frac{W}{m_o} \right|^{m_o+\delta_o}$$

$$= \exp\left[-(m_o+\delta_o)\left(\sum_{r=1}^{\infty} \frac{\operatorname{tr} W^r}{r \, m_o^r}\right)\right]$$

$$= \exp(-\operatorname{tr} W)\left[1 - \frac{1}{m_o}\left(\delta_o \operatorname{tr} W + \frac{1}{2} \operatorname{tr} W^2\right)\right.$$

$$- \frac{1}{m_o^2}\left\{\left(\frac{\delta_o}{2} \operatorname{tr} W^2 + \frac{1}{3} \operatorname{tr} W^3\right)\right.$$

$$\left.\left. - \frac{1}{2}\left(\delta_o \operatorname{tr} W + \frac{1}{2} \operatorname{tr} W^2\right)^2\right\}\right]$$

$$+ O(m_o^{-3}) \tag{4.18}$$

and

$$\tilde{C}_\kappa(P^2) m_o^k = \tilde{C}_\kappa\left(\frac{W}{m_o}\right) m_o^k = \tilde{C}_\kappa(W) \tag{4.19}$$

since the zonal polynomial $\tilde{C}_\kappa(M)$ is homogeneous of degree k. Furthermore, the use of formula for the Bernoulli

polynomial

$$B_n(x+g) = \sum_{r=0}^{n} \binom{n}{r} B_{n-r}(x) g^r \tag{4.20}$$

shows

$$A_1^*(k) = A_1^* + [(2\delta_o+1)\ k+\tilde{a}_1(\kappa)] \tag{4.21}$$

$$C_{01}^*(k) = \frac{C_{01}}{s_o} - \frac{1}{2s_o}\ [(2(a_o+\nu)-1)\ k + k^2] \tag{4.22}$$

$$Q_2^*(k) = Q_2^* - (\delta_o^2+\delta_o+\tfrac{1}{6})k + \tfrac{1}{2}\ (2\delta_o+1)^2\ k^2$$

$$- \tfrac{1}{2}\ (2\delta_o+1)\ \tilde{a}_1(\kappa) + (2\delta_o+1)k\tilde{a}_1(\kappa) + \tfrac{1}{2}\ \tilde{a}_1^2(\kappa) - \tfrac{1}{6}\ \tilde{a}_2(\kappa) \tag{4.23}$$

$$C_{02}^*(k) = C_{02}^* + \frac{(a_o+\nu)^2}{s_o^2}k + \frac{1}{2s_o^2}\ [(\nu+a_o+1)^2 + (a_o+\nu)]\ k(k-1)$$

$$+ \frac{6(a_o+\nu)+8}{12s_o^2}k(k-1)\ (k-2) + \frac{1}{8s_o^2}k\,(k-1)\ (k-2)\ (k-3) \tag{4.24}$$

$$C_{11}^*(k) = - \frac{a_o+\nu}{s_o} - \frac{1}{2s_o}\ [(2(a_o+\nu)+1)\ k + k^2] \tag{4.25}$$

where

$$A_1^* = C_{01} = 0,\ Q_2^* + C_{02}^* = 0 \tag{4.26}$$

as in Eq. (3.42), (3.50), by suitably chosen a_o, δ_o, s_o in Eq. (3.40), (3.41), (3.17) and $\tilde{a}_1(\kappa)$, $\tilde{a}_2(\kappa)$ are defined in

Eq. (2.15). Also

$$R_1(k) = Q_1(k) - C_{01}(k) \tag{4.27}$$

$$R_2(k) = Q_2(k) - C_{02}(k) - (Q_1(k) - C_{01}(k))C_{11}(k). \tag{4.28}$$

Now define

$$\beta_\theta(s_o m_o + a_o, \nu + j) = \exp(-\theta) \sum_k \frac{\theta^k}{k!} I_x(s_o m_o + a_o, \nu + j + k) \tag{4.29}$$

with $I_x(s_o m_o + a_o, \nu+j+k)$ as defined in Eq. (4.7) and

$$\theta = \text{tr } W. \tag{4.30}$$

Substitution of Eqs. (4.21) - (4.28) in Eq. (4.6), and application of Eqs. (2.18) - (2.21) over < gives the following expression for the non-null asymptotic distribution of u up to order m_o^{-2}:

$$\begin{aligned}
&\text{Prob } (u \le x) \\
&= \beta_\theta(s_o m_o + a_o, \nu) \\
&+ \frac{1}{m_o} \Big\{ \beta_\theta(s_o m_o + a_o, \nu)(-\delta_o \text{ tr } W - \tfrac{1}{2} \text{ tr } W^2) \\
&+ \beta_\theta(s_o m_o + a_o, \nu+1)\left(2\delta_o - \frac{a_o + \nu}{s_o}\right) \text{ tr } W \\
&+ \beta_\theta(s_o m_o + a_o, \nu+2) \left[\left(-\delta_o + \frac{a_o+\nu}{s_o}\right) \text{ tr } W + \text{tr } W^2 - \frac{1}{2s_o} (\text{tr } W)^2\right] \\
&+ \beta_\theta(s_o m_o + a_o, \nu+3) \left[\frac{1}{2s_o} (\text{tr } W)^2 - \tfrac{1}{2} \text{ tr } W^2\right] \Big\}
\end{aligned}$$

$$+ \frac{1}{m_o^2} \Bigg\{ \beta_\theta(s_o m_o + a_o, \nu) \left[\frac{\delta_o^2}{2} (\mathrm{tr}\ W)^2 - \frac{\delta_o}{2}\ \mathrm{tr}\ W^2 - \frac{1}{3}\ \mathrm{tr}\ W^3 \right.$$

$$\left. + \frac{\delta_o}{2}\ \mathrm{tr}\ W\ \mathrm{tr}\ W^2 + \frac{1}{8} (\mathrm{tr}\ W^2)^2 \right]$$

$$+ \beta_\theta(s_o m_o + a_o, \nu+1) \left[\left(\delta_o - \frac{a_o+\nu}{s_o}\right)^2 \mathrm{tr}\ W - \left(2\delta_o^2 - \frac{\delta_o(a_o+\nu)}{s_o}\right) (\mathrm{tr}\ W)^2 \right.$$

$$\left. - \left(\delta_o - \frac{a_o+\nu}{2s_o}\right) \mathrm{tr}\ W\ \mathrm{tr}\ W^2 \right]$$

$$+ \beta_\theta(s_o m_o + a_o, \nu+2) \left[\left(-2\delta_o^2 + \frac{\delta_o}{s_o} + \frac{4\delta_o(a_o+\nu)}{s_o} - \frac{(a+\nu)}{s_o^2} - \frac{2(a_o+\nu)^2}{s_o^2}\right) \mathrm{tr}\ W \right.$$

$$+ \left(3\delta_o - \frac{2(a_o+\nu)}{s_o} - \frac{1}{s_o}\right) \mathrm{tr}\ W^2 + \left(\frac{1}{2s_o^2}((\nu+a_o+1)^2 \right.$$

$$\left. + (\nu+a_o)) - \frac{3\delta_o(a_o+\nu)}{s_o} - \frac{2\delta_o}{s_o} + 3\delta_o^2 + \frac{1}{2} \right)$$

$$\times (\mathrm{tr}\ W)^2 + \frac{\delta_o}{2s_o} (\mathrm{tr}\ W)^3 - \left(\frac{\delta_o}{2} + \frac{a_o+\nu}{2s_o}\right) \mathrm{tr}\ W\ \mathrm{tr}\ W^2$$

$$\left. - \frac{1}{2} (\mathrm{tr}\ W^2)^2 + \frac{1}{4s_o}\ \mathrm{tr}\ W^2\ (\mathrm{tr}\ W)^2 \right]$$

$$+ \beta_\theta(s_o m_o + a_o, \nu+3) \left[\left(\delta_o^2 + \frac{(a_o+\nu)^2 + (a_o+\nu)}{s_o^2} - \frac{2\delta_o(a_o+\nu)}{s_o} - \frac{\delta_o}{s_o} \right) \mathrm{tr}\ W \right.$$

$$+ \left(\frac{7(a_o+\nu)}{2s_o} + \frac{5}{2s_o} - 4\delta_o \right) \mathrm{tr}\ W^2 + \frac{5}{3}\ \mathrm{tr}\ W^3$$

$$- \left(\frac{2(a_o+\nu)^2 + 3(a_o+\nu)+1}{2s_o^2} - \frac{3\delta_o(a_o+\nu)}{s_o} \right.$$

$$- \frac{4\delta_o}{s_o} + 2\delta_o^2 + 1\Big) \quad (\mathrm{tr}\ W)^2$$

$$+ \left(\frac{a_o+\nu}{2s_o^2} + \frac{2}{3s_o^2} - \frac{3\delta_o}{2s_o}\right) \quad (\mathrm{tr}\ W)^3 + \Big(\frac{5\delta_o}{2}$$

$$- \frac{a_o+\nu+2}{s_o}\Big) \ \mathrm{tr}\ W\ \mathrm{tr}\ W^2 - \frac{1}{4s_o}\ \mathrm{tr}\ W^2\ (\mathrm{tr}\ W^2)$$

$$+ \frac{1}{4}\ (\mathrm{tr}\ W^2)^2\Big]$$

$$+ \beta_\theta(s_o m_o + a_o, \nu+4)\left[\left(\frac{3}{2}\ \delta_o - \frac{3(a_o+\nu+1)}{2s_o}\right) \mathrm{tr}\ W^2 - 2\ \mathrm{tr}\ W^3\right.$$

$$+ \left(\frac{(a_o+\nu)^2 + 4(a_o+\nu)+2}{2s_o^2} - \frac{\delta_o(a_o+\nu+2)}{s_o} + \frac{\delta_o^2+1}{2}\right) \quad (\mathrm{tr}\ W)^2$$

$$- \left(\frac{a_o+\nu}{s_o^2} + \frac{3}{2s_o} - \frac{3\delta_o}{2s_o}\right) \quad (\mathrm{tr}\ W)^3 + \Big(\frac{3(a_o+\nu)+7}{2s_o}$$

$$- 2\delta_o\Big) \ \mathrm{tr}\ W\ \mathrm{tr}\ W^2 - \frac{1}{2s_o}\ \mathrm{tr}\ W^2\ (\mathrm{tr}\ W)^2$$

$$\left. + \frac{1}{2}\ (\mathrm{tr}\ W^2)^2 + \frac{1}{8s_o^2}\ (\mathrm{tr}\ W)^4\right]$$

$$+ \beta_\theta(s_o m_o + a_o, \nu+5)\left[\frac{2}{3}\ \mathrm{tr}\ W^3 + \left(\frac{5}{6s_o^2} + \frac{a_o+\nu}{2s_o^2} - \frac{\delta_o}{2s_o}\right) (\mathrm{tr}\ W)^3\right.$$

$$+ \left(\frac{\delta_o}{2} - \frac{a_o+\nu+3}{2s_o}\right) \mathrm{tr}\ W\ \mathrm{tr}\ W^2 + \frac{3}{4s_o}\ \mathrm{tr}\ W^2\ (\mathrm{tr}\ W)^2$$

$$\left. - \frac{1}{2}\ (\mathrm{tr}\ W^2)^2 - \frac{1}{4s_o^2}\ (\mathrm{tr}\ W)^4\right]$$

$$+ \beta_\theta(s_o m_o + a_o, \nu+6) \left[-\frac{1}{4s_o} \operatorname{tr} W^2 (\operatorname{tr} W)^2 + \frac{1}{8} (\operatorname{tr} W^2)^2 + \frac{1}{8s_o^2} (\operatorname{tr} W)^4 \right] \Bigg\}$$

$$+ O(m_o^{-3}) \tag{4.31}$$

When the alternative hypotheses are close to the null hypotheses, the accuracy obtained by using the first two terms in the asymptotic expressions is sufficient for practical purposes. When the alternative hypotheses deviate from the null hypothesis significantly, we require higher order terms to obtain very good accuracy.

5. NULL ASYMPTOTIC DISTRIBUTION OF THE TEST STATISTIC FOR MULTIPLE HOMOGENEITY OF COVARIANCE MATRICES

Let $\underset{\sim}{Z}_1,\ldots \underset{\sim}{Z}_q$ be independent complex p-variate normal variables with mean vectors $\underset{\sim}{\mu}_1,\ldots,\underset{\sim}{\mu}_q$ and covariance matrices $\Sigma_1,\ldots,\Sigma_q$, respectively. Also, Let $\underset{\sim}{Z}_{ij}(j=1,\ldots,N_i)$ be j^{th} independent observation on $\underset{\sim}{Z}_i$. Let $H_o = \bigcap_{j=1}^{d} H_{oj}$ where $H_{oj}(j=1,\ldots,d)$ is given by

$$H_{oj}:\Sigma_{q^*_{j-1}+1}=\ldots=\Sigma_{q^*_j} \tag{5.1}$$

and

$$q^*_j = q_1+\ldots+q_j,\ q^*_o = 0,\ q^*_1 = q_1 \text{ and } q^*_d = q. \tag{5.2}$$

The modified likelihood ratio statistic for testing H_o is given by

$$\lambda = \prod_{i=1}^{q} |A_i/n_i|^{n_i} \Big(\prod_{j=1}^{d} \Big| \sum_{i=q^*_{j-1}+1}^{q^*_j} A_i/n^*_j \Big|^{n^*_j} \Big)^{-1} \tag{5.3}$$

where

$$n_i = N_i - 1,\quad n^*_j = \sum_{i=q^*_{j-1}+1}^{q^*_j} n_i, \tag{5.4}$$

$$A_i = \sum_{j=1}^{N_i} (\underset{\sim}{Z}_{ij}-\underset{\sim}{Z}_{i.})(\bar{\underset{\sim}{Z}}_{ij}-\bar{\underset{\sim}{Z}}_{i.})',\ \underset{\sim}{Z}_{i.} = \frac{1}{N_i}\sum_{j=1}^{N_i}\underset{\sim}{Z}_{ij} \tag{5.5}$$

where $\bar{\underset{\sim}{Z}}$ denotes the complex conjugate.

Let $w = \lambda^{1/n}$ where $n = \sum_{i=1}^{q} n_i$ and define $\gamma_i = \frac{n_i}{n}, n = m + \delta$

where δ is a covergent factor to be determined. Thus $n_i = (m+\delta)\,\gamma_i$, and $\gamma_j^* = \sum_{i=q_{j-1}^*+1}^{q_j^*} \gamma_i$, $u = w^{1/s}$, where s is also a convergent factor. The h^{th} moment of u is

$$E(u^h) = \left\{ \prod_{\alpha=1}^{d} n_\alpha^{*\,p\frac{h}{s}\gamma_\alpha^*} \Big/ \prod_{\alpha=1}^{q} n_\alpha^{p\frac{h}{s}\gamma_\alpha} \right\} \times$$

$$\prod_{i=1}^{p} \prod_{\alpha=1}^{d} \{ \prod_{g=q_{\alpha-1}^*+1}^{q_\alpha^*} \frac{\Gamma(\gamma_g(m+\delta)(1+\frac{h}{ns})-i+1)}{\Gamma(\gamma_g(m+\delta)-i+1)}$$

$$\frac{\Gamma(\gamma_\alpha^*(m+\delta)-i+1)}{\Gamma(\gamma_\alpha^*(m+\delta)(1+\frac{h}{ns})-i+1)} . \tag{5.6}$$

The modified likelihood ratio test statistic λ and its moments are given in Krishnaiah, Lee and Chang (1976). Using Mellin inverse transform on $E(u^h)$, and following the same lines as in Section 3, we have the following expression for the c.d.f. of u:

$$\text{Prob } (u \leq x) = I_x(sm+a,\nu) + \sum_{i=1}^{\infty} \frac{1}{m^i} G_i \tag{5.7}$$

where

$$I_x(c,d) = \frac{\Gamma(c)\Gamma(d)}{\Gamma(c+d)} \int_0^x u^{c-1}(1-u)^{d-1} d\,u \tag{5.8}$$

$$G_i = \sum_{j=0}^{i} R_{i-j}\; I_x(sm+a,\nu+i-j)\left(\sum_{\ell=0}^{j} \frac{Q_\ell^*\; C_{i-j,j-\ell}}{s^{i-\ell}}\right) \tag{5.9}$$

$$\nu = \frac{p^2}{2}\,(q-d) \tag{5.10}$$

$$A_r = \sum_{i=1}^{p} \sum_{\alpha=1}^{d} \frac{(-1)^r s^r}{r(r+1)} \left[\frac{B_{r+1}(\gamma_\alpha^* \delta - i + 1)}{\gamma_\alpha^{*r}} - \sum_{g=q_{\alpha-1}^*+1}^{q_\alpha^*} \frac{B_{r+1}(\gamma_g \delta - i + 1)}{\gamma_g^r} \right] \tag{5.11}$$

$$Q_r = \frac{1}{r} \sum_{\ell=1}^{r} \ell \, A_\ell Q_{r-\ell} ; \quad Q_o = 1 \tag{5.12}$$

$$A_{ij} = \frac{(-1)^j}{j(j+1)} [B_{j+1}(\nu+a+i) - B_{j+1}(a)] \tag{5.13}$$

$$C_{ij} = \frac{1}{j} \sum_{\ell=1}^{j} \ell \, A_{i\ell} \, C_{i,j-\ell} ; \quad C_{i0} = 1 \tag{5.14}$$

$$\sum_{j=0}^{i} R_{i-j} \, C_{i-j,j} = Q_i ; \quad R_0 = 1 \tag{5.15}$$

$$A_r^* = - \frac{A_r}{s^r} \tag{5.16}$$

$$Q_r^* = \frac{1}{r} \sum_{\ell=1}^{r} \ell \, A_\ell^* \, Q_{r-\ell}^* ; \quad Q_0^* = 1 \tag{5.17}$$

$$A_{jr}^* = \frac{A_{jr}}{s^r} \tag{5.18}$$

$$C^*_{jr} = \frac{1}{r} \sum_{\ell=1}^{r} \ell \; A^*_{j\ell} \; C^*_{j,r-\ell} = \frac{C_{jr}}{s^r}; \; C^*_{j0} = 1 \tag{5.19}$$

where $B_r(x)$ is the Bernoulli polynomial.

Now, let

$$a_o = \frac{1-\nu}{2} \tag{5.20}$$

$$\eta = \sum_{\alpha=1}^{d} \left\{ \sum_{g=q^*_{\alpha-1}+1}^{q^*_\alpha} \frac{1}{\gamma_g} - \frac{1}{\gamma^*_\alpha} \right\} \tag{5.21}$$

$$\delta_o = \frac{\eta(2p^2-1)}{6p(q-d)} \tag{5.22}$$

and let s_o be the positive root of

$$s^2 = \frac{(1-\nu^2)\nu}{24A^*_2} . \tag{5.23}$$

Then the asymptotic expression for c.d.f. of u up to m_0^{-3}, where $m_0 = n - \delta_o$ is

$$\begin{aligned} \text{Prob } (u \leq x) &= I_x(s_o m_o + a_o, \nu) \\ &+ \frac{1}{m_0^3} \frac{R_3}{s_o^3} [I_x(s_o m_o + a_o, \nu + 3) \\ &- I_x(s_o m_o + a_o, \nu)] + 0\ (m_o^{-4}) \end{aligned} \tag{5.24}$$

Table 2 gives a comparison of the accuracy of the approximation by taking the first term in (5.24) with the approximation obtained by taking the first two terms in (5.24). In the table, the constant $\tilde{w}$ is defined by $c = \{\exp(-\tilde{w}/2)\}^{1/n}$ where $P[\tilde{w} \leq c] = \alpha$ and the values of $\tilde{w}$ are taken from the tables of Krishnaiah, Lee and Chang (1976). The value of $P[w \leq c]$ is denoted by α_1 or α_2 according as one term or two terms in (5.24) are used. Also, $q_1 = \ldots = q_d = q/d$ and $n_i = n_0$ for $i=1,2,\ldots,q$, $n = qn_0$.

TABLE 2

Significance Level Associated with the Asymptotic Expression for the Likelihood Ratio Test for the Multiple Homogeneity of the Covariance Matrices

n_o	q	d	p=3, α=0.05			p=4, α=0.05		
			$\tilde{w}$	α_1	α_2	$\tilde{w}$	α_1	α_2
6	3	1	37.24	.0499	.0500	67.84	.0490	.0493
6	5	1	64.10	.0497	.0499	118.15	.0475	.0486
10	3	1	33.21	.0500	.0500	56.45	.0499	.0500
10	5	1	57.85	.0499	.0499	100.19	.0497	.0499
20	3	1	30.85	.0499	.0500	50.67	.0499	.0499
20	5	1	54.13	.0500	.0500	90.92	.0500	.0500
10	6	3	46.97	.0501	.0500	81.73	.0501	.0500
10	6	2	58.66	.0499	.0499	102.21	.0498	.0499
20	6	3	43.22	.0500	.0500	72.32	.0500	.0500
20	6	2	54.49	.0500	.0500	91.76	.0500	.0500
30	6	3	42.12	.0500	.0500	69.73	.0500	.0500
30	6	2	53.26	.0500	.0500	88.85	.0500	.0500

The likelihood ratio test statistic λ given by Eq. (5.3) can be expressed as

$$\lambda = \lambda_1 \ldots \lambda_d \tag{5.25}$$

where

$$\lambda_j = \frac{\prod_{i=q_{j-1}^*+1}^{q_j^*} |A_i/n_i|^{n_i}}{\left| \sum_{i=q_{j-1}^*+1}^{q_j^*} A_i/n_j^* \right|^{n_j^*}} \tag{5.26}$$

and λ_j is the likelihood ratio statistic for testing H_{oj}. An alternative procedure to test H_o is given below. We accept or reject H_{oj}, according as

$$\lambda_j \gtrless c_\alpha$$

where

$$\begin{aligned} P[\lambda_j \geq c_\alpha;\ j=1,\ldots,d|H_o] \\ = \prod_{j=1}^{d} P[\lambda_j \geq c_\alpha|H_{oj}] = (1-\alpha) \end{aligned} \tag{5.27}$$

The c.d.f. of $\lambda_1^{1/(s_1 n_1^*)}$ can be obtained by replacing d, q and n with 1, q_1 and n_1^* respectively in Eq. (5.24); the value of s_1 is obtained by taking the square root of the right side of (5.23) after replacing d and q with 1 and q_1 respec-

tively. We can similarly obtain the c.d.f. of $\lambda_j^{1/(s_j n_j^*)}$ $(j=2,\ldots,d)$ and determine the values of $s_2,\ldots,s_d$. So, the value of c_λ in (3.22) can be determined approximately.

The test procedure discussed in this section is useful in testing the hypothesis that the diagonal blocks in the covariance matrix of a complex multivariate normal are equal when the off diagonal blocks are null matrices.

6. NONNULL ASYMPTOTIC DISTRIBUTION OF THE TEST STATISTIC FOR EQUALITY OF TWO COVARIANCE MATRICES

Here we shall consider the case for $d = 1$ and $q = 2$ in the framework of Section 5. The likelihood ratio criterion for testing the hypothesis H_o: $\Sigma_1=\Sigma_2$ against the alternative $H_1:\Sigma_1\neq\Sigma_2$ is based on the statistic

$$\lambda = \frac{n^{pn}}{n_1^{pn_1}\, n_2^{pn_2}} \frac{|A_1|^{n_1}|A_2|^{n_2}}{|A|^n} \tag{6.1}$$

where $A=A_1+A_2, A_i$ is as defined in Eq.(5.5), and $n = n_1+n_2$. Let

$$n = m_o + \delta_o,\ n_1/n = \gamma_1, n_2/n = \gamma_2,\ w = \lambda^{1/n} \tag{6.2}$$

where δ_o is defined in Eq. (5.22) and assume that

$$\begin{aligned} &\text{(i)}\quad (I - \Sigma_1^{-1}\Sigma_2) = \frac{W}{\gamma_1 m_o};\quad W \text{ is fixed as } m_o\to\infty \\ &\text{(ii)}\quad 0 < \lim \gamma_i < 1;\ i=1,2 \end{aligned} \tag{6.3}$$

The non-null h^{th} moment of $u=w^{1/s_o}$ is

$$\begin{aligned} E(u^h) = {} & \left(\frac{n}{n_1^{\gamma_1}\, n_2^{\gamma_2}}\right)^{\frac{ph}{s_o}} \frac{\tilde{\Gamma}_p\left(n_2(\frac{h}{ns_o}+1)\right)}{\tilde{\Gamma}_p(n_1)\ \tilde{\Gamma}_p(n_2)} |\Omega|^{-n_1} \\ & \times \sum_k \sum_\kappa \frac{\tilde{C}_\kappa(I-\Omega^{-1})}{k!} \frac{\tilde{\Gamma}_p(n,k)\tilde{\Gamma}_p(n_1(\frac{h}{ns_o}+1),\kappa)}{\tilde{\Gamma}_p\left(n(\frac{h}{ns_o}+1),\kappa\right)} \end{aligned} \tag{6.4}$$

where $\kappa = (k_1, k_2, \ldots, k_p)$ is a partition of k and $k_1 \geq k_2 \geq \ldots \geq k_p \geq 0$, and s_o is the square root of the right side of (5.23) and $\Omega = \Sigma_2^{-1}\Sigma_1$, while $\tilde{\Gamma}_p(b)$ is defined in Eq. (2.8), $\tilde{\Gamma}_p(b,\kappa)$ in Eq. (2.9).

The non-null distribution is obtained by using inverse Mellin transformation and following the same lines as in Section 3. The c.d.f. of u is given by

$$\text{Prob } (u \leq x) = |\Omega|^{-n_1} \sum_k \sum_\kappa \frac{\tilde{C}_\kappa(I-\Omega^{-1})}{k!} m_o^k \left\{1 + \sum_{r=1}^{\infty} \frac{Q_r^*(k)}{m_o^r}\right\}$$

$$\times \gamma_1^k \sum_{\alpha=0}^{\infty} R_\alpha(k)\ I_x(s_o m_o + a_o, \nu+\alpha)(m_o s_o)^{-\alpha} \left[1 + \sum_{r=1}^{\infty} \frac{C^*_{jr}}{m_o^r}\right] \tag{6.5}$$

where $C^*_{jr} = \frac{C_{jr}}{s_o^r}$ is defined in Eq. (5.19). $a_o = \frac{1-\nu}{2}$ is defined in Eq. (5.20) with $\nu = \frac{p^2}{2}$ as defined in Eq. (5.10). Also, we have

$$\sum_{j=0}^{i} R_{i-j}(k)\ C_{i-j,j} = Q_i(k),\quad R_0(k) = 1 \tag{6.6}$$

$$Q_r(k) = \frac{1}{r} \sum_{\ell=1}^{r} \ell A_\ell(k)\ Q_{r-\ell}(k),\quad Q_0(k) = 1 \tag{6.7}$$

and

$$A_r(k) = \frac{(-1)^r s_o^r}{r(r+1)} \sum_{i=1}^{p} \left\{ B_{r+1}(\delta_o + k_i - i + 1) - \frac{B_{r+1}(\gamma_2\delta_o - i + 1)}{\gamma_2^r} - \frac{B_{r+1}(\gamma_1\delta_o + k_i - i + 1)}{\gamma_1^r} \right\} \tag{6.8}$$

$$Q_r^*(k) = \frac{1}{r} \sum_{\ell=1}^{r} \ell \, A_\ell^*(k) \, Q_{r-\ell}^*(k), \quad Q_0^*(k) = 1 \tag{6.9}$$

$$\begin{aligned} A_r^*(k) = \frac{(-1)^r}{r(r+1)} \sum_{i=1}^{p} \Bigg\{ & \frac{B_{r+1}(\gamma_1\delta_o - i+1)}{\gamma_1^r} + \frac{B_{r+1}(\gamma_2\delta_o - i+1)}{\gamma_2^r} \\ & - B_{r+1}(\delta_o + k_i - i+1) \Bigg\} \end{aligned} \tag{6.10}$$

From Eq. (6.3), we have

$$\Omega^{-1} = I - \frac{W}{\gamma_1 m_o}$$

$$\begin{aligned} |\Omega^{-1}|^{n_1} &= \left|I - \frac{W}{\gamma_1 m_o}\right|^{\gamma_1(m_o+\delta_o)} \\ &= \exp\left[-\gamma_1(m_o+\delta_o) \left(\sum_{i=1}^{\infty} \frac{\mathrm{tr}\, W^i}{i(\gamma_1 m_o)^i} \right) \right] \\ &= \exp\,(-\mathrm{tr}\, W)\, \Big[1 - \frac{1}{m_o}(\delta_o \mathrm{tr}\, W + \frac{1}{2\gamma_1}\, \mathrm{tr}\, W^2) \\ &\qquad - \frac{1}{m_o^2}\{ (\frac{\delta_o}{2\gamma_1} \mathrm{tr}\, W^2 + \frac{1}{3\gamma_1^2} \mathrm{tr}\, W^3) \\ &\qquad - \frac{1}{2}(\delta_o \mathrm{tr}\, W + \frac{1}{2\gamma_1} \mathrm{tr}\, W^2)^2 \} \Big] \\ &\qquad + O(m_o^{-3}) \end{aligned} \tag{6.11}$$

Also

$$\tilde{C}_\kappa(I-\Omega^{-1})\, \gamma_1^k m_o^k = \tilde{C}_\kappa\left(\frac{W}{\gamma_1 m_o}\right) \gamma_1^k m_o^k = \tilde{C}_\kappa(W). \tag{6.12}$$

Now using Eq. (4.20) for the formula of the Bernoulli polynomials, we get

$$A_1(k) = \frac{(-1)s_o}{2} \sum_{i=1}^{p} \left\{ B_2(\delta_o + k_i - i + 1) - \frac{B_2(\gamma_2\delta_o - i + 1)}{\gamma_2} - \frac{B_2(\gamma_1\delta_o + k_i - i + 1)}{\gamma_1} \right\}$$

$$= A_1 + \frac{(-1)s_o}{2}\,(1 - \frac{1}{\gamma_1})(k + \tilde{a}_1(\kappa)) \tag{6.13}$$

$$A_2(k) = A_2 + \frac{s_o}{6}\left[(1 - \frac{1}{\gamma_1^2})\,(\frac{\tilde{a}_2(\kappa)}{2} + \frac{k}{2} + \frac{3}{2}\,\tilde{a}_1(\kappa)) \right.$$

$$\left. + \,(1 - \frac{1}{\gamma_1})\,(3\delta_o k + 3\delta_o \tilde{a}_1(\kappa)) \right] \tag{6.14}$$

$$Q_1^*(k) = A_1^*(k) = \frac{-1}{2} \sum_{i=1}^{p} \left[\frac{B_2(\gamma_1\delta_o - i + 1)}{\gamma_1} + \frac{B_2(\gamma_2\delta_o - i + 1)}{\gamma_2} \right.$$

$$\left. - \,B_2(\delta_o + k_i - i + 1) \right] \tag{6.15}$$

$$= A_1^* + \frac{1}{2}\,\{(2\delta_o - 1)\,k + \tilde{a}_1(\kappa)\}$$

$$Q_2^*(k) = \frac{1}{2}(A_1^*(k)\; Q_1^*(k) + 2\; A_2^*(k))$$

$$= Q_2^* + \left[\frac{1}{8}(2\delta_o + 1)^2\; k^2 + \frac{\tilde{a}_1^2(\kappa)}{8} + \frac{(2\delta_o + 1)\,k\,\tilde{a}_1(\kappa)}{4} \right.$$

$$\left. - \frac{1}{2}(\delta_o^2 + \delta_o + \frac{1}{6})\,k - \frac{1}{2}\,(\delta_o + \frac{1}{2})\tilde{a}_1(\kappa) - \frac{\tilde{a}_2(\kappa)}{12} \right] \tag{6.16}$$

where A_1, A_2, A_1^*, Q_2^* are defined in Section 5, and $\tilde{a}_1(\kappa)$, $\tilde{a}_1(\kappa)$ are defined in Eq. (2.15). By suitably chosen a_o, δ_o and s_o as in Eq. (5.20), (5.22) and (5.23), we get

$$A_1^* = C_{01} = 0, \; Q_2^* + C_{02}^* = Q_2^* + \frac{C_{02}}{s_o^2} = 0 \qquad (6.17)$$

Furthermore, from Eq. (6.6) - (6.8), we get

$$\begin{aligned} Q_1(k) &= A_1(k) \\ Q_2(k) &= \tfrac{1}{2}(A_1(k)\, Q_1(k) + 2\, A_2(k)) \\ R_1(k) &= Q_1(k) - C_{01} \\ R_2(k) &= Q_2(k) - C_{02} - (Q_1(k) - C_{01})C_{11} \end{aligned} \qquad (6.18)$$

If we now substitute these identities in Eq. (6.5), apply Eqs. (2.10) - (2.14) for the summation of zonal polynomials, and neglect the terms of higher order of m_o, we obtain the following expression for asymptotic non-null distribution.

$$\mathrm{Prob}(u \leq x) = I_x(s_o m_o + a_o, \nu) + \frac{1}{m_o} d_1 [I_x(s_o m_o + a_o, \nu) - I_x(s_o m_o + a_o, \nu+1)]$$

$$+ \frac{1}{m_o^2} \sum_{i=1}^{3} \alpha_i\, I_x(s_o m_o + a_o, \nu+i-1) + O(m_o^{-3}) \qquad (6.19)$$

where $d_1 = \frac{1}{2}\left(1 - \frac{1}{\gamma_1}\right) \operatorname{tr} W^2$

$$\alpha_1 = \delta_o d_1 + \frac{1}{2} d_1^2 + \frac{1}{3}\left(1 - \frac{1}{\gamma_1^2}\right) \operatorname{tr} W^3$$

$$\alpha_2 = -\frac{1}{2}\left(4\delta_o - \frac{\nu+1}{s_o}\right) d_1 - d_1^2 - \left(1 - \frac{1}{\gamma_1}\right)\left[\operatorname{tr} W^3 + \frac{1}{2}(\operatorname{tr} W)^2\right]$$

$$\alpha_3 = \left(\delta_o - \frac{\nu+1}{2s_o} + 1\right) d_1 + \frac{1}{2} d_1^2 + \left(\frac{2}{3} - \frac{1}{\gamma_1} + \frac{1}{3\gamma_1^2}\right) \operatorname{tr} W^3$$

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